

Local calibrations for minimizers of the Mumford-Shah functional with rectilinear discontinuity sets

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Abstract

Using a calibration method, we prove that, if w is a function which satisfies all Euler conditions for the Mumford-Shah functional on a two-dimensional open set Ω , and the discontinuity set of w is a segment connecting two boundary points, then for every point (x_0, y_0) of Ω there exists a neighbourhood U of (x_0, y_0) such that w is a minimizer of the Mumford-Shah functional on U with respect to its own boundary values on ∂U .

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1 Introduction

This paper deals with Dirichlet minimizers of the Mumford-Shah functional (see [5] and [6])

$$\int_{\Omega} |\nabla u(x, y)|^2 dx dy + \mathcal{H}^1(S_u), \quad (1.1)$$

where Ω is a bounded open subset of \mathbf{R}^2 with a Lipschitz boundary, \mathcal{H}^1 is the one-dimensional Hausdorff measure, S_u is the set of essential discontinuity points of the unknown function u , while ∇u denotes its approximate gradient (see [2] or [3]).

A *Dirichlet minimizer* of (1.1) in Ω is a function w which belongs to the space $SBV(\Omega)$ of special functions of bounded variation in Ω (see [2] or [3]) and satisfies the inequality

$$\int_{\Omega} |\nabla w(x, y)|^2 dx dy + \mathcal{H}^1(S_w) \leq \int_{\Omega} |\nabla u(x, y)|^2 dx dy + \mathcal{H}^1(S_u)$$

for every function $u \in SBV(\Omega)$ with the same trace as w on $\partial\Omega$.

Suppose that w is a Dirichlet minimizer of (1.1) in Ω and that S_w is a regular curve. Then the following equilibrium conditions are satisfied (see [5] and [6]):

- (a) w is harmonic on $\Omega \setminus S_w$;
- (b) the normal derivative of w vanishes on both sides of S_w ;
- (c) the curvature of S_w is equal to the difference of the squares of the tangential derivatives of w on both sides of S_w .

Elementary examples show that conditions (a), (b), and (c) are not sufficient for the Dirichlet minimality of w .

In this paper we prove that, if S_w is a straight line segment connecting two points of $\partial\Omega$, and the tangential derivatives $\partial_{\tau} w$ and $\partial_{\tau}^2 w$ of w do not vanish on both sides of S_w , then (a), (b), and (c) imply that every point (x_0, y_0) in Ω has an open neighbourhood U such that w is a Dirichlet minimizer of (1.1) in U . In other words, under our assumptions, conditions (a), (b), and (c) are also sufficient for the Dirichlet minimality in small domains. We hope that our proof will be useful in the future to achieve the same result without our special assumptions on S_w .

The proof is obtained by using the calibration method adapted in [1] to the functional (1.1). We construct an explicit calibration for w in the cylinder $U \times \mathbf{R}$, where U is a suitable neighbourhood of (x_0, y_0) . This construction is elementary when $(x_0, y_0) \notin S_w$ (see [1]), so we consider only the case $(x_0, y_0) \in S_w$.

The plan of the paper is the following. In Section 2 we fix the notations and we recall the main result of [1]. In Theorem 3.1 we consider the special case of the function

$$w(x, y) := \begin{cases} x & \text{if } y > 0, \\ -x & \text{if } y < 0, \end{cases}$$

and give in full detail the first example of a calibration for a discontinuous function which is not locally constant. In Theorem 3.2 we adapt the same construction to the function

$$w(x, y) := \begin{cases} x + 1 & \text{if } y > 0, \\ x & \text{if } y < 0. \end{cases}$$

In Section 4 we consider the general case of a function w satisfying (a), (b), and (c) and with $S_w = \{(x, y) \in \Omega : y = 0\}$. If S_w is connected, only two situations are possible:

$$\partial_x w(x, 0+) = -\partial_x w(x, 0-) \quad \text{on } S_w, \quad (1.2)$$

$$\partial_x w(x, 0+) = \partial_x w(x, 0-) \quad \text{on } S_w. \quad (1.3)$$

The former case (1.2) is studied in Theorem 4.1 by a suitable change of variables and by adding two new parameters to the construction used in Theorem 3.1. The minor changes for the case (1.3) are considered in Theorem 4.2.

2 Preliminary results

Let Ω be a bounded open subset of \mathbf{R}^2 with a Lipschitz boundary and let

$$\Omega_0 = \{(x, y) \in \Omega : y \neq 0\}, \quad S = \{(x, y) \in \Omega : y = 0\}.$$

For every vector field $\varphi : \Omega \times \mathbf{R} \rightarrow \mathbf{R}^2 \times \mathbf{R}$ we define the maps $\varphi^x, \varphi^y, \varphi^z : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ by

$$\varphi(x, y, z) = (\varphi^x(x, y, z), \varphi^y(x, y, z), \varphi^z(x, y, z)).$$

We shall consider the collection \mathcal{F} of all piecewise C^0 vector fields $\varphi : \Omega \times \mathbf{R} \rightarrow \mathbf{R}^2 \times \mathbf{R}$ with the following property: there exists a finite number g_1, \dots, g_k of functions in $C^1(\overline{\Omega})$ such that the sets

$$A_i := \{(x, y, z) : (x, y) \in \Omega, g_i(x, y) < z < g_{i+1}(x, y)\}$$

are nonempty and $\varphi \in C^1(\overline{A_i}, \mathbf{R}^2 \times \mathbf{R})$ for $i = 0, \dots, k$, where we put $g_0 = -\infty$ and $g_{k+1} = +\infty$. Therefore, the discontinuity set of a vector field in \mathcal{F} is contained in a finite number of regular surfaces.

Let $w \in C^1(\Omega_0)$ be a function such that $\int_{\Omega_0} |\nabla w|^2 dx dy < +\infty$. The upper trace of w on S is denoted by $w(x, 0+)$, and the lower trace by $w(x, 0-)$. Therefore, the approximate upper and lower limits $w^+(x, 0)$ and $w^-(x, 0)$ are given by

$$w^+(x, 0) = \max\{w(x, 0+), w(x, 0-)\} \quad \text{and} \quad w^-(x, 0) = \min\{w(x, 0+), w(x, 0-)\}.$$

A *calibration* for w is a bounded vector field $\varphi \in \mathcal{F}$ which is continuous on the graph of w and satisfies the following properties:

- (a) $\operatorname{div} \varphi = 0$ in the sense of distributions in $\Omega \times \mathbf{R}$;
- (b) $(\varphi^x(x, y, z))^2 + (\varphi^y(x, y, z))^2 \leq 4\varphi^z(x, y, z)$ at every continuity point (x, y, z) of φ ;
- (c) $(\varphi^x, \varphi^y)(x, y, w(x, y)) = 2\nabla w(x, y)$ and $\varphi^z(x, y, w(x, y)) = |\nabla w(x, y)|^2$ for every $(x, y) \in \Omega_0$;
- (d) $\left(\int_{t_1}^{t_2} \varphi^x(x, y, z) dz\right)^2 + \left(\int_{t_1}^{t_2} \varphi^y(x, y, z) dz\right)^2 \leq 1$ for every $(x, y) \in \Omega$ and for every $t_1, t_2 \in \mathbf{R}$;
- (e) $\int_{w^-(x, 0)}^{w^+(x, 0)} \varphi^x(x, 0, z) dz = 0$ and $\int_{w^-(x, 0)}^{w^+(x, 0)} \varphi^y(x, 0, z) dz = 1$ for every $(x, 0) \in S$.

The following theorem is proved in [1].

Theorem 2.1 *If there exists a calibration φ for w , then w is a Dirichlet minimizer of the Mumford-Shah functional (1.1) in Ω .*

If Ω is a circle with centre on the x -axis, and $w \in C^1(\Omega_0)$ with $\int_{\Omega_0} |\nabla w|^2 dx dy < +\infty$, then w satisfies the Euler conditions (a), (b), and (c) if and only if w has one of the following forms:

$$w(x, y) = \begin{cases} u(x, y) & \text{if } y > 0, \\ -u(x, y) + c_1 & \text{if } y < 0, \end{cases} \quad (2.1)$$

or

$$w(x, y) = \begin{cases} u(x, y) + c_2 & \text{if } y > 0, \\ u(x, y) & \text{if } y < 0, \end{cases} \quad (2.2)$$

where $u \in C^1(\Omega)$ is harmonic with normal derivative vanishing on S and c_1, c_2 are real constants. For our purposes, it is enough to consider the case $c_1 = 0$ in (2.1) and $c_2 = 1$ in (2.2).

3 A model case

In this section we consider in (2.1) and in (2.2) the particular function $u(x, y) = x$ and we deal with the minimality of the functions

$$w(x, y) := \begin{cases} x & \text{if } y > 0, \\ -x & \text{if } y < 0, \end{cases} \quad (3.1)$$

and

$$w(x, y) := \begin{cases} x + 1 & \text{if } y > 0, \\ x & \text{if } y < 0. \end{cases} \quad (3.2)$$

The aim of the study of these simpler cases (but we will see that they involve the main difficulties) is to clarify the ideas of the general construction.

Theorem 3.1 *Let $w : \mathbf{R}^2 \rightarrow \mathbf{R}$ be the function defined by*

$$w(x, y) := \begin{cases} x & \text{if } y > 0, \\ -x & \text{if } y < 0. \end{cases}$$

Then every point $(x_0, y_0) \neq (0, 0)$ has an open neighbourhood U such that w is a Dirichlet minimizer in U of the Mumford-Shah functional (1.1).

Proof. The result follows by Theorem 4.1 of [1] if $y_0 \neq 0$. We consider now the case $y_0 = 0$, assuming for simplicity that $x_0 > 0$. We will construct a local calibration of w near $(x_0, 0)$. Let us fix $\varepsilon > 0$ such that

$$0 < \varepsilon < \frac{x_0}{10}, \quad 0 < \varepsilon < \frac{1}{32}. \quad (3.3)$$

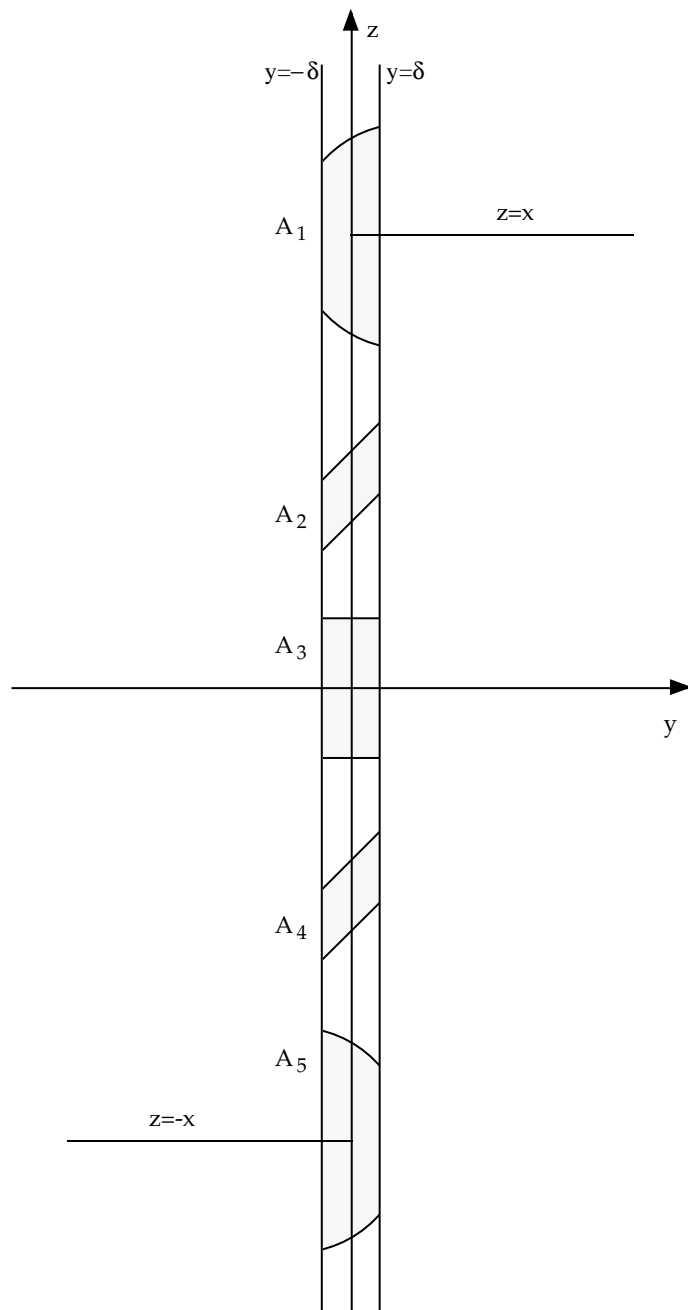


Figure 1: Section of the sets A_1, \dots, A_5 at $x = \text{constant}$.

For $0 < \delta < \varepsilon$ we consider the open rectangle

$$U := \{(x, y) \in \mathbf{R}^2 : |x - x_0| < \varepsilon, |y| < \delta\}$$

and the following subsets of $U \times \mathbf{R}$ (see Fig. 1)

$$\begin{aligned} A_1 &:= \{(x, y, z) \in U \times \mathbf{R} : x - \alpha(y) < z < x + \alpha(y)\}, \\ A_2 &:= \{(x, y, z) \in U \times \mathbf{R} : b + \kappa(\lambda)y < z < b + \kappa(\lambda)y + h\}, \\ A_3 &:= \{(x, y, z) \in U \times \mathbf{R} : -h < z < h\}, \\ A_4 &:= \{(x, y, z) \in U \times \mathbf{R} : -b + \kappa(\lambda)y - h < z < -b + \kappa(\lambda)y\}, \\ A_5 &:= \{(x, y, z) \in U \times \mathbf{R} : -x - \alpha(-y) < z < -x + \alpha(-y)\}, \end{aligned}$$

where

$$\begin{aligned} \alpha(y) &:= \sqrt{4\varepsilon^2 - (\varepsilon - y)^2}, \\ h &:= \frac{x_0 - 3\varepsilon}{4}, \quad \kappa(\lambda) := \frac{\lambda}{4} - \frac{1}{\lambda}, \quad b := 2h + \kappa(\lambda)\delta, \quad \lambda := \frac{1 - 4\varepsilon}{2h}. \end{aligned}$$

We will assume that

$$\delta < \frac{x_0 - 3\varepsilon}{8|\kappa(\lambda)|}, \quad (3.4)$$

so that the sets A_1, \dots, A_5 are pairwise disjoint.

For every $(x, y, z) \in U \times \mathbf{R}$, let us define the vector $\varphi(x, y, z) = (\varphi^x, \varphi^y, \varphi^z)(x, y, z) \in \mathbf{R}^3$ as follows:

$$\left\{ \begin{array}{ll} \left(\frac{2(\varepsilon - y)}{\sqrt{(\varepsilon - y)^2 + (z - x)^2}}, \frac{-2(z - x)}{\sqrt{(\varepsilon - y)^2 + (z - x)^2}}, 1 \right) & \text{if } (x, y, z) \in A_1, \\ \left(0, \lambda, \frac{\lambda^2}{4} \right) & \text{if } (x, y, z) \in A_2, \\ (f(y), 0, 1) & \text{if } (x, y, z) \in A_3, \\ \left(0, \lambda, \frac{\lambda^2}{4} \right) & \text{if } (x, y, z) \in A_4, \\ \left(\frac{-2(\varepsilon + y)}{\sqrt{(\varepsilon + y)^2 + (z + x)^2}}, \frac{2(z + x)}{\sqrt{(\varepsilon + y)^2 + (z + x)^2}}, 1 \right) & \text{if } (x, y, z) \in A_5, \\ (0, 0, 1) & \text{otherwise,} \end{array} \right.$$

where

$$f(y) := -\frac{1}{h} \left(\int_0^{\alpha(y)} \frac{\varepsilon - y}{\sqrt{t^2 + (\varepsilon - y)^2}} dt - \int_0^{\alpha(-y)} \frac{\varepsilon + y}{\sqrt{t^2 + (\varepsilon + y)^2}} dt \right).$$

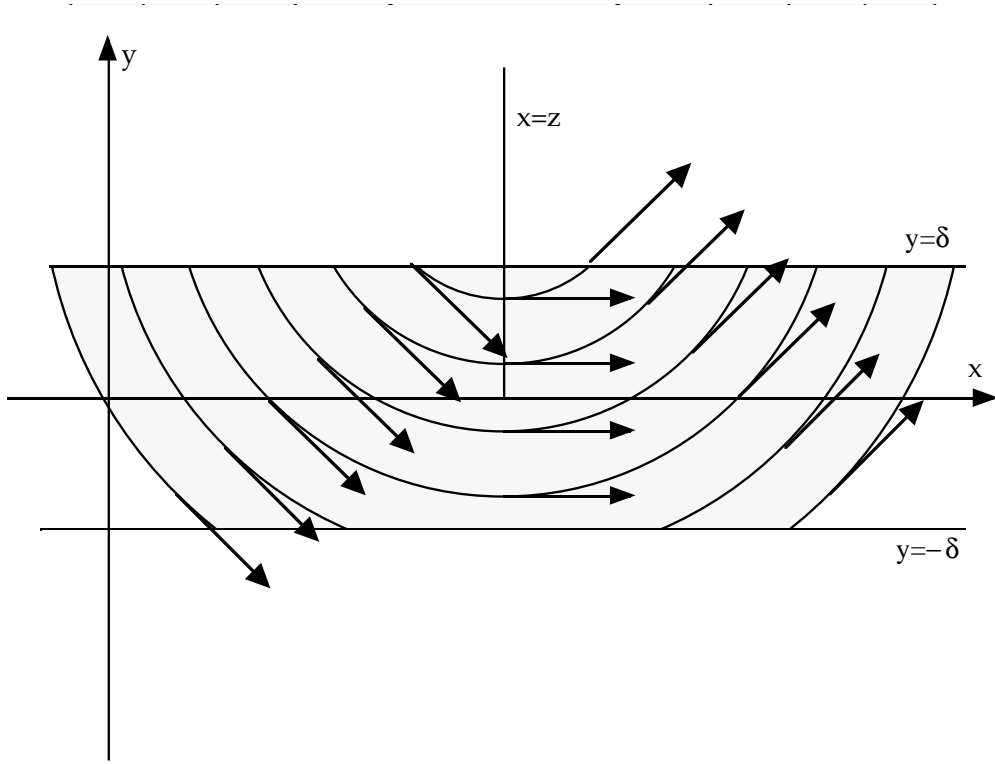


Figure 2: Section of the set A_1 at $z = \text{constant}$.

Note that $A_1 \cup A_5$ is an open neighbourhood of $\text{graph}(w) \cap (U \times \mathbf{R})$. The purpose of the definition of φ in A_1 and A_5 (see Fig. 2) is to provide a divergence free vector field satisfying condition (c) of Section 2 and such that

$$\begin{aligned} \varphi^y(x, 0, z) &> 0 & \text{for } |z| < x, \\ \varphi^y(x, 0, z) &< 0 & \text{for } |z| > x. \end{aligned}$$

These properties are crucial in order to obtain (d) and (e) simultaneously.

The rôle of A_2 and A_4 is to give the main contribution to the integral in (e). To explain this fact, suppose, for a moment, that $\varepsilon = 0$; in this case we would have $A_1 = A_5 = \emptyset$ and

$$\int_{-x}^x \varphi^y(x, 0, z) dz = 1,$$

so that the y -component of equality (e) would be satisfied.

The purpose of the definition of φ in A_3 is to correct the x -component of φ , in order to obtain (d).

We shall prove that, for a suitable choice of δ , the vector field φ is a calibration for w in the rectangle U .

Note that for a given $z \in \mathbf{R}$ we have

$$\partial_x \varphi^x(x, y, z) + \partial_y \varphi^y(x, y, z) = 0 \quad (3.5)$$

for every (x, y) such that $(x, y, z) \in A_1 \cup A_5$. This implies φ is divergence free in $A_1 \cup A_5$. Moreover $\operatorname{div} \varphi = 0$ in the other sets A_i , and the normal component of φ is continuous across ∂A_i : the choice of $\kappa(\lambda)$ ensures that this property holds for ∂A_2 and ∂A_4 (see Fig. 3). Therefore φ is divergence free in the sense of distributions in $U \times \mathbf{R}$.

On the graph of w we have

$$\varphi(x, y, w(x, y)) = \begin{cases} (2, 0, 1) & \text{if } y > 0, \\ (-2, 0, 1) & \text{if } y < 0, \end{cases}$$

so condition (c) is satisfied.

Inequality (b) is clearly satisfied in all regions: the only non trivial case is A_3 , where we have, using (3.3),

$$|f(y)| \leq \frac{4(\alpha(y) + \alpha(-y))}{x_0 - 3\varepsilon} \leq \frac{8\sqrt{3}\varepsilon}{x_0 - 3\varepsilon} < 2.$$

We now compute

$$\int_{-x}^x \varphi^y(x, y, z) dz. \quad (3.6)$$

Let us fix y with $|y| < \delta$. Since $\varphi^y(x, y, z)$ depends on $z - x$, we have

$$\int_{x-\alpha(y)}^x \varphi^y(x, y, z) dz = \int_x^{x+\alpha(y)} \varphi^y(\xi, y, x) d\xi. \quad (3.7)$$

Using (3.5) and applying the divergence theorem to the curvilinear triangle

$$T = \{(\xi, \eta) \in \mathbf{R}^2 : \xi > x, \eta < y, (\varepsilon - \eta)^2 + (x - \xi)^2 < 4\varepsilon^2\}$$

(see Fig. 4), we obtain

$$\int_x^{x+\alpha(y)} \varphi^y(\xi, y, x) d\xi = \int_{-\varepsilon}^y \varphi^x(x, \eta, x) d\eta = 2(y + \varepsilon). \quad (3.8)$$

From (3.7) and (3.8), we get

$$\int_{x-\alpha(y)}^x \varphi^y(x, y, z) dz = 2(y + \varepsilon). \quad (3.9)$$

Similarly we can prove that

$$\int_{-x}^{-x+\alpha(-y)} \varphi^y(x, y, z) dz = 2(-y + \varepsilon). \quad (3.10)$$

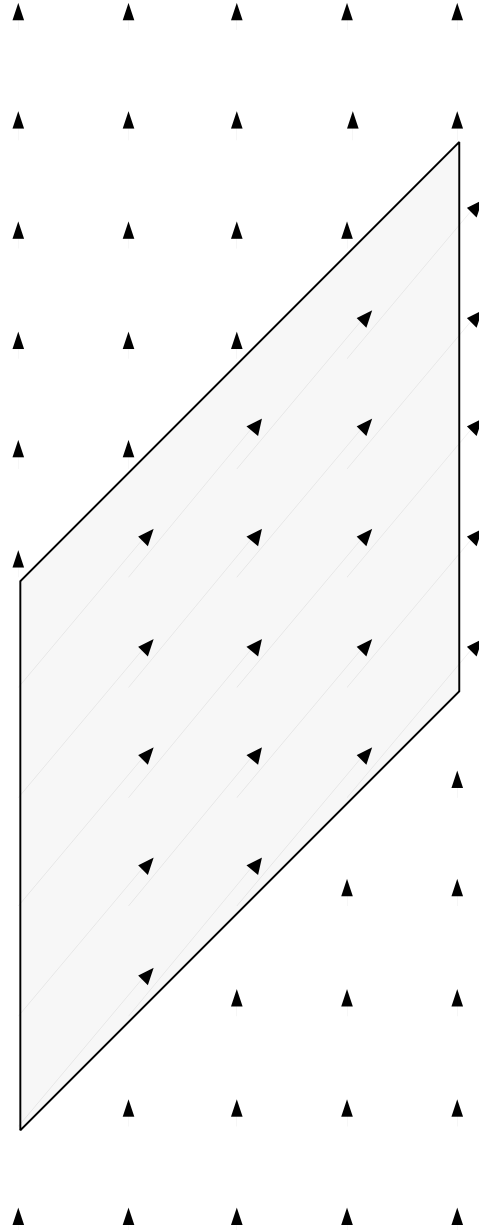


Figure 3: Section of the set A_2 at $x = \text{constant}$.

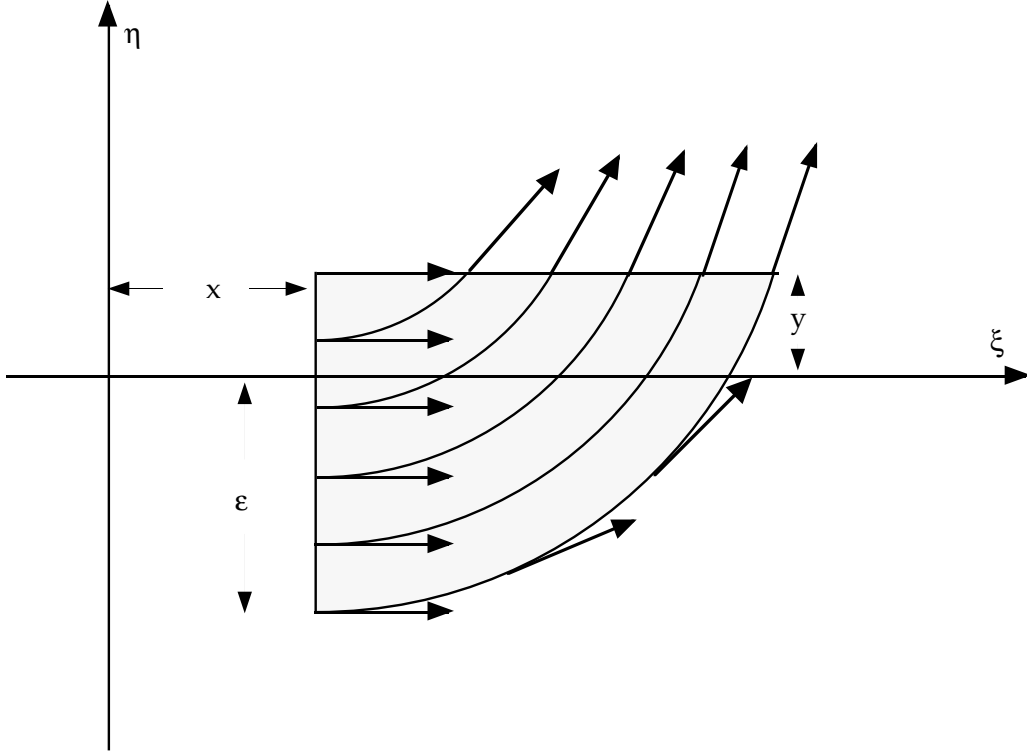


Figure 4: The curvilinear triangle T .

Using the definition of φ in A_2 , A_3 , A_4 , we obtain

$$\int_{-x}^x \varphi^y(x, y, z) dz = 1. \quad (3.11)$$

On the other hand, by the definition of f , we have immediately that

$$\int_{-x}^x \varphi^x(x, y, z) dz = 0. \quad (3.12)$$

From these equalities it follows in particular that condition (e) is satisfied on the jump set $S_w \cap U = \{(x, y) \in U : y = 0\}$.

Let us begin now the proof of (d). Let us fix $(x, y) \in U$. For every $t_1 < t_2$ we set

$$I(t_1, t_2) := \int_{t_1}^{t_2} (\varphi^x, \varphi^y)(x, y, z) dz.$$

It is enough to consider the case $-x - \alpha(-y) \leq t_1 \leq t_2 \leq x - \alpha(y)$. We can write

$$\begin{aligned} I(t_1, t_2) &= I(t_1, -x) + I(-x, x) + I(x, t_2), \\ I(t_1, -x) &= I(t_1 \wedge (-x + \alpha(-y)), -x) + I(t_1 \vee (-x + \alpha(-y)), -x + \alpha(-y)), \\ I(x, t_2) &= I(x, t_2 \vee (x - \alpha(y))) + I(x - \alpha(y), t_2 \wedge (x - \alpha(y))). \end{aligned}$$

Therefore

$$\begin{aligned} I(t_1, t_2) &= I(-x, x) + I(t_1 \wedge (-x + \alpha(-y)), -x) + I(x, t_2 \vee (x - \alpha(y))) \\ &\quad + I(t_1 \vee (-x + \alpha(-y)), t_2 \wedge (x - \alpha(y))) - I(-x + \alpha(-y), x - \alpha(y)). \end{aligned} \quad (3.13)$$

Let B be the ball of radius 4ε centred at $(0, -4\varepsilon)$. We want to prove that

$$I(x, t) \in \overline{B} \quad (3.14)$$

for every t with $x - \alpha(y) \leq t \leq x + \alpha(y)$. Let us denote the components of $I(x, t)$ by a^x and a^y . Arguing as in the proof of (3.9), we get the identity

$$a^y = 2(\varepsilon - y) - 2\sqrt{(t - x)^2 + (\varepsilon - y)^2} \leq 0. \quad (3.15)$$

As $|\varphi^x| \leq 2$, we have also

$$(a^x)^2 \leq 4(t - x)^2 = (2(\varepsilon - y) - a^y)^2 - 4(\varepsilon - y)^2.$$

From these estimates it follows that

$$(a^x)^2 + (a^y + 4\varepsilon)^2 \leq 16\varepsilon^2,$$

which proves (3.14). In the same way we can prove that

$$I(t, -x) \in \overline{B} \quad (3.16)$$

for every t with $-x - \alpha(-y) \leq t \leq -x + \alpha(-y)$.

If $f(y) \geq 0$, we define

$$C := ([0, 2hf(y)] \times [0, \frac{1}{2} - 2\varepsilon]) \cup (\{2hf(y)\} \times [0, 1 - 4\varepsilon]);$$

if $f(y) \leq 0$, we simply replace $[0, 2hf(y)]$ by $[2hf(y), 0]$. From the definition of φ in A_2, A_3, A_4 , it follows that

$$I(-x + \alpha(-y), x - \alpha(y)) = (2hf(y), 1 - 4\varepsilon) \quad (3.17)$$

and

$$I(s_1, s_2) \in C \quad (3.18)$$

for $-x + \alpha(-y) \leq s_1 \leq s_2 \leq x - \alpha(y)$. Let $D := C - (2hf(y), 1 - 4\varepsilon)$, i.e.,

$$D = ([-2hf(y), 0] \times [-1 + 4\varepsilon, -\frac{1}{2} + 2\varepsilon]) \cup (\{0\} \times [-1 + 4\varepsilon, 0]),$$

for $f(y) \geq 0$; the interval $[-2hf(y), 0]$ is replaced by $[0, -2hf(y)]$ when $f(y) \leq 0$. From (3.13), (3.11), (3.12), (3.14), (3.16), (3.17) and (3.18) we obtain

$$I(t_1, t_2) \in (0, 1) + 2\overline{B} + D. \quad (3.19)$$

As $f(0) = 0$, we can choose δ so that (3.4) is satisfied and

$$|2hf(y)| = \frac{x_0 - 3\varepsilon}{2}|f(y)| \leq \varepsilon \quad (3.20)$$

for $|y| < \delta$. It is then easy to see that, by (3.3), the set $(0, 1) + 2\overline{B} + D$ is contained in the unit ball centred at $(0, 0)$. So that (3.19) implies (d). \square

Remark. The assumption $(x_0, y_0) \neq (0, 0)$ in Theorem 3.1 cannot be dropped. Indeed, there is no neighbourhood U of $(0, 0)$ such that w is a Dirichlet minimizer of the Mumford-Shah functional in U .

To see this fact, let ψ be a function defined on the square $Q = (-1, 1) \times (-1, 1)$ satisfying the boundary condition $\psi = w$ on ∂Q and such that $S_\psi = ((-1, -1/2) \cup (1/2, 1)) \times \{0\}$. For every ε , let ψ_ε be the function defined on $Q_\varepsilon = \varepsilon Q$ by $\psi_\varepsilon(x, y) := \varepsilon\psi(x/\varepsilon, y/\varepsilon)$. Note that ψ_ε satisfies the boundary condition $\psi_\varepsilon = w$ on ∂Q_ε . Let us compute the Mumford-Shah functional for ψ_ε on Q_ε :

$$\int_{Q_\varepsilon} |\nabla \psi_\varepsilon|^2 dx dy + \mathcal{H}^1(S_{\psi_\varepsilon}) = \varepsilon^2 \int_Q |\nabla \psi|^2 dx dy + \varepsilon.$$

Since

$$\int_{Q_\varepsilon} |\nabla w|^2 dx dy + \mathcal{H}^1(S_w) = 4\varepsilon^2 + 2\varepsilon,$$

we have

$$\int_{Q_\varepsilon} |\nabla \psi_\varepsilon|^2 dx dy + \mathcal{H}^1(S_{\psi_\varepsilon}) < \int_{Q_\varepsilon} |\nabla w|^2 dx dy + \mathcal{H}^1(S_w)$$

for ε sufficiently small.

The construction shown in the proof of Theorem 3.1 can be easily adapted to define a calibration for the function w in (3.2).

Theorem 3.2 *Let $w : \mathbf{R}^2 \rightarrow \mathbf{R}$ be the function defined by*

$$w(x, y) := \begin{cases} x + 1 & \text{if } y > 0, \\ x & \text{if } y < 0. \end{cases}$$

Then every point $(x_0, y_0) \in \mathbf{R}^2$ has an open neighbourhood U such that w is a Dirichlet minimizer in U of the Mumford-Shah functional (1.1).

Proof. The result follows by Theorem 4.1 of [1] if $y_0 \neq 0$. We consider now the case $y_0 = 0$; we will construct a local calibration of w near $(x_0, 0)$, using the same technique as in Theorem 3.1. We give only the new definitions of the sets A_1, \dots, A_5 and of the function φ , and leave to the reader the verification of the fact that this function is a calibration for suitable values of the involved parameters.

Let us fix $\varepsilon > 0$ such that

$$0 < \varepsilon < \frac{1}{24}, \quad 0 < \varepsilon < \frac{1}{32}. \quad (3.21)$$

For $0 < \delta < \varepsilon$ we consider the open rectangle

$$U := \{(x, y) \in \mathbf{R}^2 : |x - x_0| < \varepsilon, |y| < \delta\}$$

and the following subsets of $U \times \mathbf{R}$

$$\begin{aligned} A_1 &:= \{(x, y, z) \in U \times \mathbf{R} : x + 1 - \alpha(y) < z < x + 1 + \alpha(y)\}, \\ A_2 &:= \{(x, y, z) \in U \times \mathbf{R} : b + \kappa(\lambda)y + 3h < z < b + \kappa(\lambda)y + 4h\}, \\ A_3 &:= \{(x, y, z) \in U \times \mathbf{R} : x_0 + 3\varepsilon + 2h < z < x_0 + 3\varepsilon + 3h\}, \\ A_4 &:= \{(x, y, z) \in U \times \mathbf{R} : b + \kappa(\lambda)y < z < b + \kappa(\lambda)y + h\}, \\ A_5 &:= \{(x, y, z) \in U \times \mathbf{R} : x - \alpha(-y) < z < x + \alpha(-y)\}, \end{aligned}$$

where

$$\begin{aligned} \alpha(y) &:= \sqrt{4\varepsilon^2 - (\varepsilon - y)^2}, \\ h &:= \frac{1 - 6\varepsilon}{5}, \quad \kappa(\lambda) := \frac{\lambda}{4} - \frac{1}{\lambda}, \quad b := x_0 + 3\varepsilon + \kappa(\lambda)\delta, \quad \lambda := \frac{1 - 4\varepsilon}{2h}. \end{aligned}$$

We will assume that

$$\delta < \frac{1 - 6\varepsilon}{10|\kappa(\lambda)|}, \quad (3.22)$$

so that the sets A_1, \dots, A_5 are pairwise disjoint.

For every $(x, y, z) \in U \times \mathbf{R}$, let us define the vector $\varphi(x, y, z) \in \mathbf{R}^3$ as follows:

$$\left\{ \begin{array}{ll} \left(\frac{2(\varepsilon - y)}{\sqrt{(\varepsilon - y)^2 + (z - x - 1)^2}}, \frac{-2(z - x - 1)}{\sqrt{(\varepsilon - y)^2 + (z - x - 1)^2}}, 1 \right) & \text{if } (x, y, z) \in A_1, \\ \left(0, \lambda, \frac{\lambda^2}{4} \right) & \text{if } (x, y, z) \in A_2, \\ (f(y), 0, 1) & \text{if } (x, y, z) \in A_3, \\ \left(0, \lambda, \frac{\lambda^2}{4} \right) & \text{if } (x, y, z) \in A_4, \\ \left(\frac{2(\varepsilon + y)}{\sqrt{(\varepsilon + y)^2 + (z - x)^2}}, \frac{2(z - x)}{\sqrt{(\varepsilon + y)^2 + (z - x)^2}}, 1 \right) & \text{if } (x, y, z) \in A_5, \\ (0, 0, 1) & \text{otherwise,} \end{array} \right.$$

where

$$f(y) := -\frac{2}{h} \left(\int_0^{\alpha(y)} \frac{\varepsilon - y}{\sqrt{t^2 + (\varepsilon - y)^2}} dt + \int_0^{\alpha(-y)} \frac{\varepsilon + y}{\sqrt{t^2 + (\varepsilon + y)^2}} dt \right)$$

for every $|y| < \delta$. \square

4 The general case

In this section we denote by Ω a ball in \mathbf{R}^2 centred at $(0,0)$ and we consider as u in (2.1) and in (2.2) a generic harmonic function with normal derivative vanishing on S . We add the technical assumption that the first and second order tangential derivatives of u are not zero on S .

Theorem 4.1 *Let $u : \Omega \rightarrow \mathbf{R}$ be a harmonic function such that $\partial_y u(x,0) = 0$ for $(x,0) \in \Omega$, and let $w : \Omega \rightarrow \mathbf{R}$ be the function defined by*

$$w(x,y) := \begin{cases} u(x,y) & \text{for } y > 0, \\ -u(x,y) & \text{for } y < 0. \end{cases}$$

Assume that $u_0 := u(0,0) \neq 0$, $\partial_x u(0,0) \neq 0$, and $\partial_x^2 u(0,0) \neq 0$. Then there exists an open neighbourhood U of $(0,0)$ such that w is a Dirichlet minimizer in U of the Mumford-Shah functional (1.1).

Proof. We may assume $u(0,0) > 0$ and $\partial_x u(0,0) > 0$. We shall give the proof only for $\partial_x^2 u(0,0) > 0$, and we shall explain at the end the modification needed for $\partial_x^2 u(0,0) < 0$. Let $v : \Omega \rightarrow \mathbf{R}$ be the harmonic conjugate of u that vanishes on $y = 0$, i.e., the function satisfying $\partial_x v(x,y) = -\partial_y u(x,y)$, $\partial_y v(x,y) = \partial_x u(x,y)$, and $v(x,0) = 0$.

Consider a small neighbourhood U of $(0,0)$ such that the map $\Phi(x,y) := (u(x,y), v(x,y))$ is invertible on U and $\partial_x u > 0$ on U . We call Ψ the inverse function $(u,v) \mapsto (\xi(u,v), \eta(u,v))$, which is defined in the neighbourhood $V := \Phi(U)$ of $(u_0, 0)$. Note that, if U is small enough, then $\eta(u,v) = 0$ if and only if $v = 0$. Moreover,

$$D\Psi = \begin{pmatrix} \partial_u \xi & \partial_v \xi \\ \partial_u \eta & \partial_v \eta \end{pmatrix} = \frac{1}{|\nabla u|^2} \begin{pmatrix} \partial_x u & \partial_x v \\ \partial_y u & \partial_y v \end{pmatrix}, \quad (4.1)$$

where, in the last formula, all functions are computed at $(x,y) = \Psi(u,v)$, and so $\partial_u \xi = \partial_v \eta$, $\partial_v \xi = -\partial_u \eta$ and $\partial_u \eta(u,0) = 0$, $\partial_v \eta(u,0) > 0$. In particular, ξ and η are harmonic, and

$$\partial_u^2 \eta(u,0) = 0, \quad \partial_v^2 \eta(u,0) = 0. \quad (4.2)$$

On U we will use the coordinate system (u,v) given by Φ . By (4.1) the canonical basis of the tangent space to U at a point (x,y) is given by

$$\tau_u = \frac{\nabla u}{|\nabla u|^2}, \quad \tau_v = \frac{\nabla v}{|\nabla v|^2}. \quad (4.3)$$

For every $(u,v) \in V$, let $G(u,v)$ be the matrix associated with the first fundamental form of U in the coordinate system (u,v) , and let $g(u,v)$ be its determinant. By (4.1) and (4.3),

$$g = ((\partial_u \eta)^2 + (\partial_v \eta)^2)^2 = \frac{1}{|\nabla u(\Psi)|^4}. \quad (4.4)$$

We set $\gamma(u, v) := \sqrt[4]{g(u, v)}$.

The calibration $\varphi(x, y, z)$ on $U \times \mathbf{R}$ will be written as

$$\varphi(x, y, z) = \frac{1}{\gamma^2(u(x, y), v(x, y))} \phi(u(x, y), v(x, y), z). \quad (4.5)$$

We will adopt the following representation for $\phi : V \times \mathbf{R} \rightarrow \mathbf{R}^3$:

$$\phi(u, v, z) = \phi^u(u, v, z)\tau_u + \phi^v(u, v, z)\tau_v + \phi^z(u, v, z)e_z, \quad (4.6)$$

where e_z is the third vector of the canonical basis of \mathbf{R}^3 , and τ_u, τ_v are computed at the point $\Psi(u, v)$. We now reformulate the conditions of Section 2 in this new coordinate system. It is known from Differential Geometry (see, e.g., [4, Proposition 3.5]) that, if $X = X^u\tau_u + X^v\tau_v$ is a vector field on U , then the divergence of X is given by

$$\operatorname{div} X = \frac{1}{\gamma^2}(\partial_u(\gamma^2 X^u) + \partial_v(\gamma^2 X^v)). \quad (4.7)$$

Using (4.3), (4.4), (4.5), (4.6), and (4.7) it turns out that φ is a calibration if the following conditions are satisfied:

- (a) $\partial_u \phi^u + \partial_v \phi^v + \partial_z \phi^z = 0$ for every $(u, v, z) \in V \times \mathbf{R}$;
- (b) $(\phi^u(u, v, z))^2 + (\phi^v(u, v, z))^2 \leq 4\phi^z(u, v, z)$ for every $(u, v, z) \in V \times \mathbf{R}$;
- (c) $\phi^u(u, v, \pm u) = \pm 2$, $\phi^v(u, v, \pm u) = 0$, and $\phi^z(u, v, \pm u) = 1$ for every $(u, v) \in V$;
- (d) $\left(\int_{t_1}^{t_2} \phi^u(u, v, z) dz\right)^2 + \left(\int_{t_1}^{t_2} \phi^v(u, v, z) dz\right)^2 \leq \gamma^2(u, v)$ for every $(u, v) \in V$, $t_1, t_2 \in \mathbf{R}$;
- (e) $\int_{-u}^u \phi^u(u, 0, z) dz = 0$ and $\int_{-u}^u \phi^v(u, 0, z) dz = \gamma(u, 0)$ for every $(u, 0) \in V$.

Given suitable parameters $\varepsilon > 0$, $h > 0$, $\lambda > 0$, that will be chosen later, and assuming

$$V = \{(u, v) : |u - u_0| < \delta, |v| < \delta\}, \quad (4.8)$$

with $\delta < \varepsilon$, we consider the following subsets of $V \times \mathbf{R}$

$$\begin{aligned} A_1 &:= \{(u, v, z) \in V \times \mathbf{R} : u - \alpha(v) < z < u + \alpha(v)\}, \\ A_2 &:= \{(u, v, z) \in V \times \mathbf{R} : 3h + \beta(u, v) < z < 3h + \beta(u, v) + 1/\lambda\}, \\ A_3 &:= \{(u, v, z) \in V \times \mathbf{R} : -h < z < h\}, \\ A_4 &:= \{(u, v, z) \in V \times \mathbf{R} : -3h + \beta(u, v) - 1/\lambda < z < -3h + \beta(u, v)\}, \\ A_5 &:= \{(u, v, z) \in V \times \mathbf{R} : -u - \alpha(-v) < z < -u + \alpha(-v)\}, \end{aligned}$$

where

$$\alpha(v) := \sqrt{4\varepsilon^2 - (\varepsilon - v)^2},$$

and β is a suitable smooth function satisfying $\beta(u, 0) = 0$, which will be defined later. It is easy to see that, if ε and h are sufficiently small, while λ is sufficiently large, then the sets A_1, \dots, A_5 are pairwise

disjoint, provided δ is small enough. Moreover, since $\gamma(u, 0) = \partial_v \eta(u, 0) > 0$, by continuity we may assume that

$$\gamma(u, v) > 128\varepsilon \quad \text{and} \quad \partial_v \eta(u, v) > 8\varepsilon \quad (4.9)$$

for every $(u, v) \in V$.

For $(u, v) \in V$ and $z \in \mathbf{R}$ the vector $\phi(u, v, z)$ introduced in (4.5) is defined as follows:

$$\left\{ \begin{array}{ll} \frac{2(\varepsilon - v)}{\sqrt{(\varepsilon - v)^2 + (z - u)^2}} \tau_u - \frac{2(z - u)}{\sqrt{(\varepsilon - v)^2 + (z - u)^2}} \tau_v + e_z & \text{in } A_1, \\ -\lambda\sigma(u, v) \frac{v}{\sqrt{(u - a)^2 + v^2}} \tau_u + \lambda\sigma(u, v) \frac{u - a}{\sqrt{(u - a)^2 + v^2}} \tau_v + \mu e_z & \text{in } A_2, \\ f(v) \tau_u + e_z & \text{in } A_3, \\ -\lambda\sigma(u, v) \frac{v}{\sqrt{(u - a)^2 + v^2}} \tau_u + \lambda\sigma(u, v) \frac{u - a}{\sqrt{(u - a)^2 + v^2}} \tau_v + \mu e_z & \text{in } A_4, \\ -\frac{2(\varepsilon + v)}{\sqrt{(\varepsilon + v)^2 + (z + u)^2}} \tau_u + \frac{2(z + u)}{\sqrt{(\varepsilon + v)^2 + (z + u)^2}} \tau_v + e_z & \text{in } A_5, \\ e_z & \text{otherwise,} \end{array} \right.$$

where

$$a < u_0 - 11\delta, \quad \mu > 0 \quad (4.10)$$

$$f(v) := -\frac{1}{h} \left(\int_0^{\alpha(v)} \frac{(\varepsilon - v)}{\sqrt{t^2 + (\varepsilon - v)^2}} dt - \int_0^{\alpha(-v)} \frac{(\varepsilon + v)}{\sqrt{t^2 + (\varepsilon + v)^2}} dt \right),$$

$$\sigma(u, v) := \frac{1}{2} \gamma(a + \sqrt{(u - a)^2 + v^2}, 0) - 2\varepsilon. \quad (4.11)$$

We choose β as the solution of the Cauchy problem

$$\left\{ \begin{array}{l} \lambda\sigma(u, v)(-v \partial_u \beta + (u - a) \partial_v \beta) = (\mu - 1) \sqrt{(u - a)^2 + v^2}, \\ \beta(u, 0) = 0. \end{array} \right. \quad (4.12)$$

Since the line $v = 0$ is not characteristic for the equation near $(u_0, 0)$, there exists a unique solution $\beta \in C^\infty(V)$, provided V is small enough.

In the coordinate system (u, v) the definition of the field ϕ in A_1 , A_3 , and A_5 is the same as the definition of φ in the proof of Theorem 3.1. The crucial difference is in the definition on the sets A_2 and A_4 , where now we are forced to introduce two new parameters a and μ . Note that the definition given in Theorem 3.1 can be regarded as the limiting case as a tends to $+\infty$.

By direct computations it is easy to see that ϕ satisfies condition (a) on A_1 and A_5 . Similarly, the vector field

$$\left(-\frac{v}{\sqrt{(u-a)^2 + v^2}}, \frac{u-a}{\sqrt{(u-a)^2 + v^2}} \right)$$

is divergence free; since $(u-a)^2 + v^2$ is constant along the integral curves of this field, by construction the same property holds for σ , so that ϕ satisfies condition (a) in A_2 and A_4 .

In A_3 , condition (a) is trivially satisfied.

Note that the normal component of ϕ is continuous across each ∂A_i : for the region A_3 this continuity is guaranteed by our choice of β . This implies that (a) is satisfied in the sense of distributions on $V \times \mathbf{R}$.

In order to satisfy condition (b), it is enough to take the parameter μ such that

$$\frac{\lambda^2}{4} \sigma^2(u, v) \leq \mu$$

for every $(u, v) \in V$, and require that

$$|f(v)| \leq 2. \quad (4.13)$$

Since

$$|f(v)| \leq \frac{\alpha(v) + \alpha(-v)}{h} \leq \frac{4\varepsilon}{h}, \quad (4.14)$$

inequality (4.13) is true if we impose

$$2\varepsilon \leq h.$$

Looking at the definition of ϕ on A_1 and A_5 , one can check that condition (c) is satisfied.

Arguing as in the proof of (3.9), (3.10), (3.12) in Theorem 3.1, we find that for every $(u, v) \in V$

$$\begin{aligned} \int_{-u}^{-u+\alpha(-v)} \phi^u(u, v, z) dz + \int_{-h}^h \phi^u(u, v, z) dz + \int_{u-\alpha(v)}^u \phi^u(u, v, z) dz &= 0, \\ \int_{-u}^{-u+\alpha(-v)} \phi^v(u, v, z) dz + \int_{-h}^h \phi^v(u, v, z) dz + \int_{u-\alpha(v)}^u \phi^v(u, v, z) dz &= 4\varepsilon. \end{aligned}$$

Now, it is easy to see that

$$\int_{-u}^u \phi^u(u, v, z) dz = -2\sigma(u, v) \frac{v}{\sqrt{(u-a)^2 + v^2}}, \quad (4.15)$$

$$\int_{-u}^u \phi^v(u, v, z) dz = 4\varepsilon + 2\sigma(u, v) \frac{u-a}{\sqrt{(u-a)^2 + v^2}}; \quad (4.16)$$

since for $v = 0$ we have

$$\sigma(u, 0) = \frac{1}{2} \gamma(u, 0) - 2\varepsilon,$$

condition (e) is satisfied.

By continuity, if δ is small enough, we have

$$\int_{-u}^u \phi^v(u, v, z) dz > \frac{7}{8} \gamma(u, v) \quad (4.17)$$

for every $(u, v) \in V$.

From now on, we regard the pair (ϕ^u, ϕ^v) as a vector in \mathbf{R}^2 . To prove condition (d) we set

$$I_{\varepsilon, a}(u, v, s, t) := \int_s^t (\phi^u, \phi^v)(u, v, z) dz$$

for every $(u, v) \in V$, and for every $s, t \in \mathbf{R}$. We want to compare the behaviour of the functions $|I_{\varepsilon, a}|^2$ and γ^2 ; to this aim, we define the function

$$d_{\varepsilon, a}(u, v, s, t) := |I_{\varepsilon, a}(u, v, s, t)|^2 - \gamma^2(u, v).$$

We have already shown (condition (e)) that

$$d_{\varepsilon, a}(u, 0, -u, u) = 0. \quad (4.18)$$

We start by proving that, if V is sufficiently small, condition (d) holds for every $(u, v) \in V$, for t_1 close to $-u$ and t_2 close to u . Using the definition of $\phi(u, v, z)$ on A_1 and A_5 , one can compute explicitly $d_{\varepsilon, a}(u, v, s, t)$ for $|s + u| \leq \alpha(-v)$ and for $|t - u| \leq \alpha(v)$. By direct computations one obtains

$$\nabla_{v, s, t} d_{\varepsilon, a}(u, 0, -u, u) = 0 \quad (4.19)$$

for $(u, 0) \in V$.

We now want to compute the hessian matrix $\nabla_{v, s, t}^2 d_{\varepsilon, a}$ at the point $(u_0, 0, -u_0, u_0)$. By (4.11) and (4.4), after some easy computations, we get

$$\partial_v^2 \sigma(u, 0) = \frac{1}{2(u-a)} \partial_u \gamma(u, 0) = \frac{1}{2(u-a)} \partial_v \partial_u \eta(u, 0).$$

Using this equality and the explicit expression of $d_{\varepsilon, a}$ near $(u_0, 0, -u_0, u_0)$, we obtain

$$\begin{aligned} \partial_v^2 d_{\varepsilon, a}(u_0, 0, -u_0, u_0) &= -\frac{8\varepsilon}{(u_0 - a)^2} (\partial_v \eta(u_0, 0) - 4\varepsilon) + \\ &+ \frac{2}{u_0 - a} \partial_v \eta(u_0, 0) \partial_v \partial_u \eta(u_0, 0) - \partial_v^2 (\gamma^2)(u_0, 0). \end{aligned}$$

Since η and γ do not depend on a and ε , for every ε satisfying (4.9) we can find a so close to u_0 that

$$\partial_v^2 d_{\varepsilon, a}(u_0, 0, -u_0, u_0) < 0. \quad (4.20)$$

Moreover, we easily obtain that

$$\begin{aligned} \partial_t^2 d_{\varepsilon, a}(u_0, 0, -u_0, u_0) &= \partial_s^2 d_{\varepsilon, a}(u_0, 0, -u_0, u_0) = 8 - \frac{4}{\varepsilon} \partial_v \eta(u_0, 0), \\ \partial_v \partial_t d_{\varepsilon, a}(u_0, 0, -u_0, u_0) &= \partial_v \partial_s d_{\varepsilon, a}(u_0, 0, -u_0, u_0) = -\frac{4}{u_0 - a} (\partial_v \eta(u_0, 0) - 4\varepsilon), \\ \partial_t \partial_s d_{\varepsilon, a}(u_0, 0, -u_0, u_0) &= 8. \end{aligned}$$

By the above expressions, it follows that

$$\begin{aligned} & \det \begin{pmatrix} \partial_v^2 d_{\varepsilon,a} & \partial_v \partial_t d_{\varepsilon,a} \\ \partial_v \partial_t d_{\varepsilon,a} & \partial_t^2 d_{\varepsilon,a} \end{pmatrix} (u_0, 0, -u_0, u_0) = \\ & = \frac{16}{(u_0 - a)^2} \partial_v \eta(u_0, 0) (\partial_v \eta(u_0, 0) - 4\varepsilon) + \frac{c_1(\varepsilon)}{u_0 - a} + c_2(\varepsilon), \end{aligned}$$

where $c_1(\varepsilon)$, $c_2(\varepsilon)$ are two constants depending only on ε . Then, if ε satisfies (4.9), a can be chosen so close to u_0 that

$$\det \begin{pmatrix} \partial_v^2 d_{\varepsilon,a} & \partial_v \partial_t d_{\varepsilon,a} \\ \partial_v \partial_t d_{\varepsilon,a} & \partial_t^2 d_{\varepsilon,a} \end{pmatrix} (u_0, 0, -u_0, u_0) > 0. \quad (4.21)$$

At last, the determinant of the hessian matrix of $d_{\varepsilon,a}$ at $(u_0, 0, -u_0, u_0)$ is given by

$$\det \nabla_{v,s,t}^2 d_{\varepsilon,a}(u_0, 0, -u_0, u_0) = \frac{1}{u_0 - a} (\partial_v \eta(u_0, 0))^2 \partial_v \partial_u \eta(u_0, 0) (\partial_v \eta(u_0, 0) - 4\varepsilon) \frac{32}{\varepsilon^2} + c_3(\varepsilon),$$

where $c_3(\varepsilon)$ is a constant depending only on ε . Since, by (4.1),

$$\partial_v \partial_u \eta(u_0, 0) = -\frac{\partial_x^2 u(0, 0)}{(\partial_x u(0, 0))^3},$$

given ε satisfying (4.9), we can choose a so close to u_0 that

$$\det \nabla_{v,s,t}^2 d_{\varepsilon,a}(u_0, 0, -u_0, u_0) < 0. \quad (4.22)$$

By (4.20), (4.21), and (4.22), we can conclude that, by a suitable choice of the parameters, the hessian matrix of $d_{\varepsilon,a}$ (with respect to v, s, t) at $(u_0, 0, -u_0, u_0)$ is negative definite. This fact, with (4.18) and (4.19), allows us to state the existence of a constant $\tau > 0$ such that

$$d_{\varepsilon,a}(u, v, s, t) < 0 \quad (4.23)$$

for $|s + u_0| < \tau$, $|t - u_0| < \tau$, $(u, v) \in V$, $v \neq 0$, provided V is sufficiently small. So, condition (d) is satisfied for $|t_1 + u_0| < \tau$ and $|t_2 - u_0| < \tau$. We can assume $\delta < \tau < \alpha(v)$ for every $(u, v) \in V$.

From now on, since at this point the parameters ε , a have been fixed, we simply write I instead of $I_{\varepsilon,a}$. We now study the more general case $|t_1 + u| < \alpha(-v)$ and $|t_2 - u| < \alpha(v)$.

Let us set

$$m_1(u, v) := \max \{ |I(u, v, s, t)| : |s + u| \leq \alpha(-v), |t - u| \leq \alpha(v), |t - u_0| \geq \tau \}.$$

By the definition of A_1, \dots, A_5 , for $\rho = \alpha(\delta) + \delta$ we have $(\phi^u, \phi^v) = 0$ on $(V \times [u_0 - \rho, u_0 + \rho]) \setminus A_1$ and $(V \times [-u_0 - \rho, -u_0 + \rho]) \setminus A_5$. This implies that

$$m_1(u, v) := \max \{ |I(u, v, s, t)| : |s + u_0| \leq \rho, \tau \leq |t - u_0| \leq \rho \}$$

for $(u, v) \in V$. The function m_1 , as supremum of a family of continuous functions, is lower semicontinuous. Moreover, m_1 is also upper semicontinuous; indeed, suppose, by contradiction, that there exist

two sequences (u_n) , (v_n) converging respectively to u , v , such that $(m_1(u_n, v_n))$ converges to a limit $l > m_1(u, v)$; then, there exist (s_n) , (t_n) such that

$$|s_n + u_n| \leq \alpha(-v_n), \quad |t_n - u_n| \leq \alpha(v_n), \quad |t_n - u_0| \geq \tau, \quad (4.24)$$

and $m_1(u_n, v_n) = |I(u_n, v_n, s_n, t_n)|$. Up to subsequences, we can assume that (s_n) , (t_n) converge respectively to s , t such that, by (4.24),

$$|s + u| \leq \alpha(-v), \quad |t - u| \leq \alpha(v), \quad |t - u_0| \geq \tau;$$

hence, we have that

$$m_1(u, v) \geq |I(u, v, s, t)| = \lim_{n \rightarrow \infty} |I(u_n, v_n, s_n, t_n)| = l > m_1(u, v),$$

which is impossible. Therefore, m_1 is continuous.

Let B be the open ball of radius 4ε centred at $(0, -4\varepsilon)$. Arguing as in (3.14), we can prove that

$$I(u, v, u, t) \in B \quad (4.25)$$

whenever $0 < |t - u| \leq \alpha(v)$. In the same way we can prove that

$$I(u, v, s, -u) \in B \quad (4.26)$$

for $0 < |s + u| \leq \alpha(-v)$. We can write

$$I(u, v, s, t) = I(u, v, s, -u) + I(u, v, -u, u) + I(u, v, u, t). \quad (4.27)$$

So, for $|s + u| \leq \alpha(-v)$, $|t - u| \leq \alpha(v)$, and $|t - u_0| \geq \tau$, by (4.26), (4.15), (4.16), and (4.25), we obtain that

$$I(u, 0, s, t) \in (0, \gamma(u, 0)) + B + \overline{B},$$

hence, by (4.9), $I(u, 0, s, t)$ belongs to the open ball of radius $\gamma(u, 0)$ centred at $(0, 0)$, and so, $m_1(u, 0) < \gamma(u, 0)$. By continuity, if V is small enough,

$$m_1(u, v) < \gamma(u, v) \quad (4.28)$$

for every $(u, v) \in V$.

Analogously, we define

$$m_2(u, v) := \max \{ |I(u, v, s, t)| : |s + u| \leq \alpha(-v), |s + u_0| \geq \tau, |t - u| \leq \alpha(v) \}.$$

Arguing as in the case of m_1 , we can prove that, if V is small enough,

$$m_2(u, v) < \gamma(u, v) \quad (4.29)$$

for every $(u, v) \in V$.

By (4.28), (4.29), and (4.23), we can conclude that $I(u, v, t_1, t_2)$ belongs to the ball centred at $(0, 0)$ with radius $\gamma(u, v)$, for $|t_1 + u| \leq \alpha(-v)$ and $|t_2 - u| \leq \alpha(v)$. More precisely, let $E(u, v)$ be the intersection of this ball with the upper half plane bounded by the horizontal straight line passing through the point $(0, \frac{3}{4}\gamma(u, v))$: by (4.27), (4.17), (4.25), (4.26), and (4.9), we deduce that

$$I(u, v, t_1, t_2) \in E(u, v) \quad (4.30)$$

for $|t_1 + u| \leq \alpha(-v)$ and $|t_2 - u| \leq \alpha(v)$.

We can now conclude the proof of (d). It is enough to consider the case $-u - \alpha(-v) \leq t_1 \leq t_2 \leq u + \alpha(v)$. We can write

$$\begin{aligned} I(u, v, t_1, t_2) &= I(u, v, t_1 \wedge (-u + \alpha(-v)), t_2 \vee (u - \alpha(v))) + \\ &\quad + I(u, v, t_1 \vee (-u + \alpha(-v)), t_2 \wedge (u - \alpha(v))) - \\ &\quad - I(u, v, -u + \alpha(-v), u - \alpha(v)). \end{aligned} \quad (4.31)$$

By (4.30), it follows that

$$I(u, v, t_1 \wedge (-u + \alpha(-v)), t_2 \vee (u - \alpha(v))) \in E(u, v). \quad (4.32)$$

Let $C_1(u, v)$ be the parallelogram having three consecutive vertices at the points

$$(2hf(v), 0), \quad (0, 0), \quad \sigma(u, v) \frac{(-v, u - a)}{\sqrt{(u - a)^2 + v^2}},$$

let $C_2(u, v)$ be the segment with endpoints

$$(2hf(v), 0), \quad (2hf(v), 0) + 2\sigma(u, v) \frac{(-v, u - a)}{\sqrt{(u - a)^2 + v^2}},$$

and let $C(u, v) := C_1(u, v) \cup C_2(u, v)$.

From the definition of φ in A_2, A_3, A_4 , it follows that

$$I(u, v, -u + \alpha(-v), u - \alpha(v)) = (2hf(v), 0) + 2\sigma(u, v) \frac{(-v, u - a)}{\sqrt{(u - a)^2 + v^2}} \quad (4.33)$$

and

$$I(u, v, s_1, s_2) \in C(u, v) \quad (4.34)$$

for $-u + \alpha(-v) \leq s_1 \leq s_2 \leq u - \alpha(v)$. Let

$$D(u, v) := C(u, v) - (2hf(v), 0) - 2\sigma(u, v) \frac{(-v, u - a)}{\sqrt{(u - a)^2 + v^2}}.$$

From (4.31), (4.32), (4.33), and (4.34) we obtain

$$I(u, v, t_1, t_2) \in E(u, v) + D(u, v). \quad (4.35)$$

As $|v| < \delta < 10\delta < u - a$ by (4.10), the angle that the segment $C_2(u, v)$ forms with the vertical is less than $\arctan(1/10)$. Moreover, we may assume that the length $2\sigma(u, v)$ of the segment $C_2(u, v)$ is less than $\gamma(u, v)$; indeed, this is true for $v = 0$ and, by continuity, it remains true if δ is small enough. By (4.9) and (4.14), we have also that $|2hf(v)| \leq \gamma(u, v)/16$. Using these properties and simple geometric considerations, it is possible to prove that $E(u, v) + D(u, v)$ is contained in the ball with centre $(0, 0)$ and radius $\gamma(u, v)$. This concludes the proof of (d).

If $\partial_x^2 u(0, 0) < 0$, it is enough to change the definition of ϕ in the sets A_2 and A_4 , as follows:

$$\lambda\sigma(u, v) \frac{v}{\sqrt{(a-u)^2 + v^2}} \tau_u + \lambda\sigma(u, v) \frac{a-u}{\sqrt{(a-u)^2 + v^2}} \tau_v + \mu e_z,$$

where $a > u_0 + 11\delta$ and

$$\sigma(u, v) := \frac{1}{2} \gamma(a - \sqrt{(a-u)^2 + v^2}, 0) - 2\varepsilon.$$

□

Theorem 4.2 *Let $u : \Omega \rightarrow \mathbf{R}$ be a harmonic function such that $\partial_y u(x, 0) = 0$ for $(x, 0) \in \Omega$, and let $w : \Omega \rightarrow \mathbf{R}$ be the function defined by*

$$w(x, y) := \begin{cases} u(x, y) + 1 & \text{for } y > 0, \\ u(x, y) & \text{for } y < 0. \end{cases}$$

Assume that $\partial_x u(0, 0) \neq 0$ and $\partial_x^2 u(0, 0) \neq 0$. Then there exists an open neighbourhood U of $(0, 0)$ such that w is a Dirichlet minimizer in U of the Mumford-Shah functional (1.1).

Proof. We will write the calibration φ as in (4.5) and we will adopt the representation (4.6) for ϕ . We will use the same technique as in Theorem 4.1. We give only the new definitions of the sets A_1, \dots, A_5 and of the function ϕ when $\partial_x u(0, 0) > 0$ and $\partial_x^2 u(0, 0) > 0$, and leave to the reader the verification of the fact that this function is a calibration for suitable values of the involved parameters. The case $\partial_x^2 u(0, 0) < 0$ can be treated by the changes introduced at the end of Theorem 4.1.

Let $u_0 := u(0, 0)$. Given $\varepsilon > 0$, $h > 0$, $\lambda > 0$, and assuming

$$V := \{(u, v) : |u - u_0| < \delta, |v| < \delta\},$$

we consider the following subsets of $V \times \mathbf{R}$

$$\begin{aligned} A_1 &:= \{(u, v, z) \in V \times \mathbf{R} : u + 1 - \alpha(v) < z < u + 1 + \alpha(v)\}, \\ A_2 &:= \{(u, v, z) \in V \times \mathbf{R} : 5h + \beta(u, v) < z < 5h + \beta(u, v) + 1/\lambda\}, \\ A_3 &:= \{(u, v, z) \in V \times \mathbf{R} : 2h < z < 4h\}, \\ A_4 &:= \{(u, v, z) \in V \times \mathbf{R} : h + \beta(u, v) < z < h + \beta(u, v) + 1/\lambda\}, \\ A_5 &:= \{(u, v, z) \in V \times \mathbf{R} : u - \alpha(-v) < z < u + \alpha(-v)\}, \end{aligned}$$

where

$$\alpha(v) := \sqrt{4\varepsilon^2 - (\varepsilon - v)^2},$$

and β is a suitable smooth function satisfying $\beta(u, 0) = 0$, which will be defined later. For $(u, v) \in V$ and $z \in \mathbf{R}$ the vector $\phi(u, v, z)$ is defined as follows:

$$\left\{ \begin{array}{ll} \frac{2(\varepsilon - v)}{\sqrt{(\varepsilon - v)^2 + (z - u - 1)^2}} \tau_u - \frac{2(z - u - 1)}{\sqrt{(\varepsilon - v)^2 + (z - u - 1)^2}} \tau_v + e_z & \text{in } A_1, \\ -\lambda \sigma(u, v) \frac{v}{\sqrt{(u - a)^2 + v^2}} \tau_u + \lambda \sigma(u, v) \frac{u - a}{\sqrt{(u - a)^2 + v^2}} \tau_v + \mu e_z & \text{in } A_2, \\ f(v) \tau_u + e_z & \text{in } A_3, \\ -\lambda \sigma(u, v) \frac{v}{\sqrt{(u - a)^2 + v^2}} \tau_u + \lambda \sigma(u, v) \frac{u - a}{\sqrt{(u - a)^2 + v^2}} \tau_v + \mu e_z & \text{in } A_4, \\ \frac{2(\varepsilon + v)}{\sqrt{(\varepsilon + v)^2 + (z - u)^2}} \tau_u + \frac{2(z - u)}{\sqrt{(\varepsilon + v)^2 + (z - u)^2}} \tau_v + e_z & \text{in } A_5, \\ e_z & \text{otherwise,} \end{array} \right.$$

where $a < u_0 - 11\delta$, $\mu > 0$,

$$f(v) := -\frac{1}{h} \left(\int_0^{\alpha(v)} \frac{(\varepsilon - v)}{\sqrt{t^2 + (\varepsilon - v)^2}} dt + \int_0^{\alpha(-v)} \frac{(\varepsilon + v)}{\sqrt{t^2 + (\varepsilon + v)^2}} dt \right),$$

$$\sigma(u, v) := \frac{1}{2} \gamma(a + \sqrt{(u - a)^2 + v^2}, 0) - 2\varepsilon,$$

and β is the solution of the Cauchy problem (4.12). \square

References

- [1] Alberti G., Bouchitté G., Dal Maso G.: The calibration method for the Mumford-Shah functional. Preprint SISSA, Trieste, 1998.
- [2] Ambrosio L.: A compactness theorem for a new class of variational problems. *Boll. Un. Mat. It.* **3-B** (1989), 857-881.
- [3] Ambrosio L., Fusco N., Pallara D.: Special Functions of Bounded Variation and Free-Discontinuity Problems. Oxford University Press, Oxford, to appear.
- [4] Chavel I.: Riemannian Geometry - A Modern Introduction. Cambridge University Press, Cambridge, 1993.
- [5] Mumford D., Shah J.: Boundary detection by minimizing functionals, I. *Proc. IEEE Conf. on Computer Vision and Pattern Recognition (San Francisco, 1985)*.
- [6] Mumford D., Shah J.: Optimal approximation by piecewise smooth functions and associated variational problems. *Comm. Pure Appl. Math.* **42** (1989), 577-685.